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Studying Electroweak Baryogenesis using Evenisation and the Wigner Formalism

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Abstract

We derive the kinetic equation for fermions and antifermions interacting with a planar Higgs bubble wall during the electroweak phase transition using the 'evenisation' procedure and the Wigner formalism for a Lagrangian with the phase of the complex fermion mass rotated away. We obtain the energy, velocity and force for the particles in the presence of the Higgs bubble wall. Our results using both methods are in agreement. This indicates the robustness of evenisation as a method to study quantum corrections to the velocity and force for particles in the Higgs wall during the electroweak phase transition. We also derive the transport equations from the zeroth and first moment of the kinetic equation.

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I. INTRODUCTION

In models of electroweak baryogenesis, baryon asymmetry is created during the electroweak phase transition at temperatures of around 100 GeV. Since this temperature corresponds to an energy scale that is experimentally accessible, there has been considerable interest in ascertaining the details of electroweak baryogenesis models, in particular, the kinetic equation or Boltzmann equation for the transport of particles through the Higgs bubble wall in a first order electroweak phase transition.

It is well known that, in the absence of collisions, the characteristics of the Boltzmann equation are the single particle trajectory equations, i.e. if $f(\mathbf{p}, \mathbf{x}, t)$ is a distribution function for a system of particles then $\frac{df(\mathbf{p},\mathbf{x},t)}{dt} = 0$, where $\mathbf{x}$ and $\mathbf{p}$ satisfy single particle equations of motion. Expanding the total derivative using the chain rule,

$$\partial_t f + \frac{d\mathbf{x}}{dt}\partial_{\mathbf{x}} f + \frac{d\mathbf{p}}{dt}\partial_{\mathbf{p}} f \;=\; 0\,, \tag{1}$$

and $\frac{d\mathbf{x}}{dt}$ and $\frac{d\mathbf{p}}{dt}$ can be obtained for the system from the single particle equations of motion (Hamilton's equations). However, in relativistic quantum theories it is non-trivial to obtain the single particle equations of motion from the Hamiltonian because of interference between particle and antiparticle states causing Zitterbewegung. But one may obtain equations of motion that have a consistent single particle interpretation by making use of 'evenised' operators [1]. In Ref. [2] we used evenisation to calculate particle trajectories to $O(\hbar)$ in the presence of the Higgs bubble wall during the electroweak phase transition with a complex mass term given by $m_R\bar{\psi}\psi + im_I\bar{\psi}\gamma^5\psi = |m|\bar{\psi}e^{i\theta\gamma^5}\psi$. In this article we apply the procedure of evenisation to study the interaction of fermions and antifermions with the Higgs bubble wall with the phase in the mass term rotated away and with an additional term $\frac{1}{2}\partial_\mu\theta\bar{\psi}\gamma^\mu\gamma^5\psi$ in the Lagrangian. We use this to obtain the energy relation $E(\mathbf{p}, \mathbf{x})$ and the kinetic equation.

Alternatively, in Ref. [3], the energy relation, $\omega_{s\pm}(\mathbf{p}, \mathbf{x})$, and a kinetic equation in the absence of collisions for particles and antiparticles interacting with the Higgs bubble wall are obtained from the equation of motion for the Wigner function $G^<(x, k)$, which is the Wigner transform of the Wightman function $i\left\langle\bar{\psi}(x')\psi(x'')\right\rangle$. (Also see Refs. [4, 5].) In this approach, particle and antiparticle distribution functions are extracted from the Wigner function using a spectral decomposition that separates positive and negative energy states. The particle and antiparticle distribution functions are then found to satisfy a collisionless quantum corrected kinetic equation in the presence of the external force field provided by the Higgs

wall. In Ref. [3] the Lagrangian was as in Ref. [2], i.e. with a complex pseudoscalar mass term, as mentioned above. Here we apply the Wigner formalism to the rotated Lagrangian. We obtain the energy relation and kinetic equation. These results agree with those obtained using evenisation.

The aim of this article is to establish the robustness of the application of the evenisation procedure to study quantum corrections to particle trajectories. Evenisation is a convenient technique that allows us to reobtain classical expressions ($O(\hbar^0)$) for the velocity and force for a relativistic quantum system of particles[1]. However, as far as we are aware, the extension of evenisation techniques to obtain quantum corrections to the velocity and force for a system of particles was first done in Ref. [2] and shown to be consistent with the Wigner formalism. Below we work with a different Lagrangian and again find consistent results between the two methods.

The energy relation and kinetics (velocity and force) for a fermion in the bubble wall with the real mass term were first derived in Refs. [6, 7] using a WKB ansatz for the components of the spinor wave function of the fermion. Refs. [6, 7] considered states related with chirality while below we consider spin eigenstates. However our results are also compatible with the results in Refs. [6, 7] [1].

One may analyse the interaction of the particles with the background Higgs field with either the rotated or the unrotated Lagrangian. The final baryon asymmetry is associated with the net axial current generated in the wall and its outward diffusion away from the wall. While the Lagrangian is not invariant under the rotation, the axial current $\langle\bar{\psi}\gamma^\mu\gamma^5\psi\rangle$ is invariant, and so the final asymmetry produced should be the same.

The energy relation obtained for the unrotated Lagrangian with a complex mass is

$$E \;=\; \sqrt{{p_z}^2+|m|^2}-\frac{\hbar s}{2({p_z}^2+|m|^2)}|m|^2\theta' \tag{2}$$

and can not be easily recast as a mass shell condition. The energy relation for the rotated Lagrangian with a real mass is

$$E \;=\; \sqrt{{p_z}^2+m^2}-\frac{\hbar s\theta'}{2} \tag{3}$$

and can be more easily restated as a mass shell condition $(E+\frac{\hbar s\theta'}{2})^2-p_z^2=m^2$. This is

[1] Eq. (9) of Ref. [7] is missing an overall minus sign in the expression for the force in the z direction.

similar to the mass shell condition for classical systems and may make it easier to interpret the behaviour of the particles of the system.

II. KINETIC EQUATION USING EVENISATION

A first order electroweak phase transition proceeds via the formation of Higgs bubbles. As the bubbles expand they move through the ambient sea of quarks, leptons and other particles. In this section we first obtain a semi-classical expression for the single particle energy in terms of the evenised position and momentum. We then obtain expressions for the evenised velocity and force and substitute these in Eq. (1) to obtain the kinetic equation. A brief introduction to evenisation is given in Ref. [2]. Here we merely present the necessary formulae for our analysis.

Any operator $\hat{A}$ can be split into an even part $\left[\hat{A}\right]$ and into an odd part $\left\{\hat{A}\right\}$. Even and odd parts of the operators are defined by using the sign operator [1],

$$\hat{\Lambda} = \frac{\hat{H}}{\sqrt{\hat{H}^2}}\,, \tag{4}$$

where $\hat{H}$ is the Hamiltonian. The eigenvalues of the sign operator are ± 1, corresponding to particle and antiparticle states. Then

$$\left[\hat{A}\right] = \frac{1}{2}\left(\hat{A} + \hat{\Lambda}\hat{A}\hat{\Lambda}\right) \tag{5}$$

$$\left\{\hat{A}\right\} = \frac{1}{2}\left(\hat{A} - \hat{\Lambda}\hat{A}\hat{\Lambda}\right) = \frac{1}{2}[\hat{A},\hat{\Lambda}]\hat{\Lambda} \tag{6}$$

The even part of the product of two operators $\hat{A}$ and $\hat{B}$ can be written as

$$\left[\hat{A}\hat{B}\right] = \left[\hat{A}\right]\left[\hat{B}\right] + \left\{\hat{A}\right\}\left\{\hat{B}\right\} \tag{7}$$

We now apply the above to the system under consideration.

The Lagrangian describing the interaction of particles with the bubble wall can be modeled by

$$\mathcal{L} = i\bar{\psi}\,\partial\!\!\!/\,\psi + \frac{1}{2}\partial_\mu\theta\bar{\psi}\gamma^\mu\gamma^5\psi - \frac{m}{\hbar}\bar{\psi}\psi \tag{8}$$

This is a rotated form of the Lagrangian of Ref. [2] with m above equivalent to $|m|$ in Ref. [2]. The Higgs bubble will be treated as a background field which provides in the bubble wall frame a spatially varying real mass for the particles, and a term associated with the

axial current. We shall consider the limit of large bubbles when the walls can be treated as planar. The corresponding Dirac equation

$$(i\,\partial\!\!\!/ + \frac{1}{2}\partial_\mu\theta\gamma^\mu\gamma^5 - \frac{m}{\hbar})\psi = 0 \tag{9}$$

can be rewritten in the form

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi\,, \tag{10}$$

with the Hamiltonian $\hat{H}$ given by

$$\begin{aligned} \hat{H} &= \hat{\alpha}^i(\hat{p}^i + \frac{\hbar}{2}\widehat{\partial^i\theta}\hat{\gamma}^5) + \hat{\beta}\hat{m} & (11)\\ &= \hat{\alpha}^i\hat{p}^i + \frac{\hbar}{2}\widehat{\partial^i\theta}\hat{S}^i + \hat{\beta}\hat{m}\,. & (12)\end{aligned}$$

We have defined the spin operator $\hat{\mathbf{S}}$ as $\hat{\alpha}\hat{\gamma}^5$, without the standard $\hbar/2$ in the definition to be consistent with the notation in Ref. [3]; the spin operator defined here is $\hat{\boldsymbol{\Sigma}}$ with Pauli matrices along the diagonal. We have adopted the metric (1,-1,-1,-1). In this section $\hat{}$ designates an operator. We now use the method of evenisation and evenise $\hat{H}^2$. We discuss later why we evenise $\hat{H}^2$ rather than $\hat{H}$.

We first write after some algebra,

$$\hat{H}^2 = \hat{\mathbf{p}}^2 + \hat{m}^2 + \frac{\hbar^2}{4}(\widehat{\partial^i\theta}\hat{S}^i)^2 + \hat{\alpha}^i\hat{\beta}\left[\hat{p}^i,\hat{m}\right] + \hbar\widehat{\partial^i\theta}\hat{m}\hat{S}^i\hat{\beta} + \frac{\hbar}{2}\hat{\alpha}^i\hat{S}^j[\hat{p}^i,\widehat{\partial^j\theta}] + \hbar\widehat{\partial^j\theta}\hat{S}^j\hat{\alpha}^i\hat{p}^i \tag{13}$$

We work in the wall frame and take the wall to be planar in the $x-y$ plane and so $\hat{m}$ is a function of $\hat{z}$ only. Then, keeping only terms to $O(\hbar)$, the Hamiltonian can be written as

$$\hat{H}^2 = \hat{\mathbf{p}}^2 + \hat{m}^2 - \hat{S}^3\hat{\gamma}^0\hat{\gamma}^5(-i\hbar\widehat{\partial_z m}) - \hbar\hat{\theta}'\hat{m}\hat{S}^3\hat{\beta} - \hbar\hat{\theta}'\hat{p}^i\hat{S}^3\hat{\alpha}^i\,, \tag{14}$$

where $z \equiv x^3$, $\partial_z \equiv \partial/\partial\hat{x}^3$ and $\hat{\theta}' = \widehat{\partial_z\theta} = -\widehat{\partial^3\theta}$. We chose to express $\hat{\alpha}^3\hat{\beta}$ in terms of $\hat{S}^3$ as this simplifies the evenisation of the corresponding term below.

We now evenise $[\hat{H}^2]$ to order $\hbar$. The spin operator in general does not commute with the Hamiltonian. In order to simplify the problem, as in Ref.[3], we assume that we are in an inertial reference frame where x and y components of the momentum are zero and consequently $\hat{\alpha}^1$ and $\hat{\alpha}^2$ will be absent from the Hamiltonian. This will allow us to set $\{S^3\}$ to 0 later. As mentioned earlier, we wish to express $[\hat{H}^2]$ in terms of $[\hat{p}_z]$ and $[\hat{z}]$ (where $p_z \equiv p^3$). The odd part of $\hat{p}_z$ will be proportional to $[\hat{\Lambda},\hat{p}_z]$ (see Eq. (6) above) and so will be of order $\hbar$. Therefore, using Eq. (7), we can approximate $[\hat{p}_z^2]$ by $[\hat{p}_z]^2$. Similarly we

approximate $[m(\hat{z})^2]$ as $[m(\hat{z})]^2$. Furthermore, since $\{\hat{z}\}$ is $O(\hbar)$, $[m(\hat{z})] = m([\hat{z}]) + O(\hbar^2)$ as any dependence of the even operator $[m(\hat{z})]$ on the odd operator $\{\hat{z}\}$ can only appear as $\{\hat{z}\}^2$. (Expand $m(z)$ in a Taylor series about $z = 0$ and then replace z by $\hat{z} = [\hat{z}] + \{\hat{z}\}$.) The last three terms on the right hand side of Eq. (14) are $O(\hbar)$ and therfore we will use the sign operator defined upto $O(\hbar^0)$ to evenise these terms. We define the zeroth order Hamiltonian and energy as

$$\hat{H}_0 = \hat{\alpha}^3\hat{p}^3 + \hat{\beta}\hat{m} \tag{15}$$

$$E_0 = \sqrt{{p_z}^2 + m^2} \tag{16}$$

where ${p_z}^2$ and m^2 are real numbers and are expectation values of the corresponding operators in an eigenstate of definite energy and spin. With this we define the sign operator upto $O\left(\hbar^0\right)$ as follows

$$\hat{\Lambda}_0 = \frac{\hat{H}_0}{\sqrt{{p_z}^2 + m^2}} \tag{17}$$

Note that $(\hat{\Lambda}_0)^2 = 1 + O(\hbar)$ which we shall use in the derivation of evenised operators in the Appendix. We obtain

$$[\hat{\gamma}^0\hat{\gamma}^5] = O(\hbar) \tag{18}$$

$$[\hat{\beta}] = \frac{\hat{m}}{E_0}\hat{\Lambda}_0 + O(\hbar)\,. \tag{19}$$

$$[\hat{\alpha}^3] = \frac{[\hat{p}_z]}{E_0}\hat{\Lambda}_0 + O(\hbar)\,. \tag{20}$$

Now

$$[\hat{S}^3] = [\hat{\alpha^3\gamma^5}] = \hat{S}^3\,, \tag{21}$$

i.e., $\{\hat{S}^3\} = 0$ as $\hat{S}^3$ commutes with the Hamiltonian (which is true only in the chosen inertial frame). Then, keeping in mind that the odd parts of $\hat{m}, \partial_z\hat{m}, \hat{\theta}'$ and $\hat{p}^3$ that appear in the last three terms of $\hat{H}^2$ are $O(\hbar)$, we get, to $O(\hbar)$,

$$[\hat{H}^2] = [\hat{p}_z]^2 + \hat{m}^2 - \hbar\hat{\theta}'\frac{\hat{m}^2}{E_0}\hat{S}^3\hat{\Lambda}_0 - \hbar\hat{\theta}'\frac{[\hat{p}_z]^2}{E_0}\hat{S}^3\hat{\Lambda}_0\,. \tag{22}$$

where $\hat{m}$ and $\hat{\theta}'$ are now functions of $[\hat{z}]$. Replacing evenised operators $[\hat{z}]$ and $[\hat{p}_z]$ by real numbers representing their expectation values in a state of definite energy and spin, $|E, s\rangle$

for particles and $|-E,-s\rangle$ for antiparticles [2] , we deduce the corresponding expression for the energy to be

$$E^2 \;=\; {p_z}^2 + m^2 - \hbar s\theta' E_0, \tag{23}$$

where we have also replaced $\hat{S}^3$ by its eigenvalue $\pm s$ and Λ_0 by ± 1 for particle/antiparticle states. The energy relation for particles and antiparticles respectively can then be written upto order $\hbar$ as

$$E \;=\; \sqrt{{p_z}^2+m^2} - \frac{\hbar s\theta'}{2} \tag{24}$$

Note that if we had wished to obtain the energy relation by evenising $\hat{H}$ instead of evenizing $\hat{H}^2$ we would have faced a problem as we would have needed $\hat{\Lambda}$ to $O(\hbar)$ which itself requires E to $O(\hbar)$. However, since $\{\hat{H}\} = 0$, $[\hat{H}] = [\hat{H}^2]^{1/2}$. We are now able to define the sign operator upto $O(\hbar)$ as $\hat{\Lambda} = \hat{H}/E$ and can use the same to obtain the evenised velocity and force to $O(\hbar)$ for the kinetic equation.

We now obtain the kinetic equation to $O(\hbar)$. Using the chain rule for partial differentiation the kinetic equation is written as

$$\partial_t f_{s\pm} + \frac{dz}{dt}\partial_z f_{s\pm} + \frac{dp_z}{dt}\partial_{p_z} f_{s\pm} \;=\; 0\,, \tag{25}$$

where $f_{s\pm}$ are the particle and antiparticle distribution functions. We will associate dz/dt and dp_z/dt with the expectation values of $[d\hat{z}/dt]$ and $[d\hat{p}_z/dt]$ in states of definite energy and spin, as before. Implicitly we are assuming here that the form of the quantum Boltzmann equation is the same as that of the classical Boltzmann equation and that the quantum corrections are contained only in the coefficients of the equation, namely, in the expressions for the velocity and the force. The evenised expressions for the velocity and force to $O(\hbar)$ are derived in the Appendix as

$$\begin{aligned}
[d\hat{z}/dt] \;&=\; [-\frac{i}{\hbar}[\hat{z},\hat{H}]\,] \\
&=\; \left(\frac{[\hat{p}_z]}{E} - \frac{\hbar\hat{\theta}'[\hat{p}_z]\hat{S}^3\hat{\Lambda}_0}{2E_0^2}\right)\hat{\Lambda}
\end{aligned} \tag{26}$$

$$[d\hat{p}_z/dt] \;=\; [-\frac{i}{\hbar}[\hat{p}_z,\hat{H}]\,]$$

[2] In the earlier version of this paper, we had identified antiparticles of spin s with negative energy particle solutions of spin s, rather than $-s$ (see Sec. 7.1 of Ref. [1]). Hence the expressions for energy and force were different for particles and antiparticles of the same spin. Our current results reflect the P and CP violation and C conservation properties of the Lagrangian in Eq. (8).

$$= \left(-\frac{\hat{m}^{2\prime}}{2E} + \frac{\hbar \hat{m}^{2\prime}\hat{\theta}'\hat{\Lambda}_0 \hat{S}^3}{4E_0^2} + \frac{\hbar\hat{\theta}''\hat{S}^3\hat{\Lambda}_0}{2}\right)\hat{\Lambda}\,. \tag{27}$$

Substituting the eigenvalues of the above operators in the kinetic equation [3], we get

$$\partial_t f_{s\pm} + \frac{p_z}{E}\left(1 - \frac{\hbar\theta' s}{2E_0}\right)\partial_z f_{s\pm} + \left[-\frac{m^{2\prime}}{2E}\left(1 - \frac{\hbar\theta' s}{2E_0}\right) + \frac{\hbar s\theta''}{2}\right]\partial_{p_z} f_{s\pm} \;=\; 0\,. \tag{28}$$

Simplifying the above equation by using Eq. (24) to express E as $E_0[1 - \hbar\theta' s/(2E_0)]$ we get

$$\partial_t f_{s\pm} + \frac{p_z}{E_0}\partial_z f_{s\pm} + \left(-\frac{m^{2\prime}}{2E_0} + \frac{\hbar s\theta''}{2}\right)\partial_{p_z} f_{s\pm} \;=\; 0\,, \tag{29}$$

Several comments are in order here. If we apply Hamilton's equations to the energy relation in Eq. (24), i.e., $dz/dt = \partial E/\partial p_z$ and $dp_z/dt = -\partial E/\partial z$, we get expressions for dz/dt and dp_z/dt that agree with the expectation values of the corresponding evenised operators $[d\hat{z}/dt]$ and $[d\hat{p}_z/dt]$. But the real number p_z appearing above in Eq. (24) need not be the canonical momentum. It represents the expectation value of $[\hat{p}_z]$ whereas the canonical momentum would be associated with the expectation value of $\hat{p}_z$. Secondly, ignoring for now the distinction between $\hat{p}_z$ and $[\hat{p}_z]$, Ehrenfest's theorem for a relativistic system would imply $d\langle\hat{z}\rangle/dt = \langle\partial\hat{H}/\partial\hat{p}_z\rangle$ and $d\langle\hat{p}_z\rangle/dt = -\langle\partial\hat{H}/\partial\hat{z}\rangle$, and not $d\langle\hat{z}\rangle/dt = \partial\langle\hat{H}\rangle/\partial\langle\hat{p}_z\rangle$ and $d\langle\hat{p}_z\rangle/dt = -\partial\langle\hat{H}\rangle/\partial\langle\hat{z}\rangle$. Nevertheless the energy relation in Eq. (24) behaves as a quantum corrected classical Hamiltonian, i.e., one may apply Hamilton's equations to it to obtain the velocity and the force. However, in general, one can not presume that E is the quantum corrected classical Hamiltonian and indeed in Ref. [2] with the unrotated Lagrangian one can not treat $E(p_z, z)$ as the quantum corrected classical Hamiltonian with p_z as the canonical momentum.

In the literature [3, 8] there has been discussion on identifying the canonical and kinetic momentum of the particle. One may presume that since $\hat{p}_z$ is identified with the $-i\hat{\partial}_z$ operator it is conjugate to $\hat{z}$ and represents the canonical momentum. Furthermore p_z acts as a canonical momentum if the energy relation of Eq. (24) is treated as a classical Hamiltonian. Now our energy relation also agrees with with that obtained below using the Wigner formalism. But that energy relation is necessarily in terms of the kinetic momentum as the Wigner function is always written in terms of the kinetic momentum of the system (see

[3] The overall minus sign in the expectation value for the velocity and force for antiparticles can be absorbed by taking the expectation value of the momentum in antiparticle states to be $-p_z$, as for systems with constant mass, where the energy eignenstates are also momentum eigenstates.

Sec. 3 of Ref. [9]). This indicates that our real number $p_z = \langle [\hat{p}_z] \rangle$ in the energy relation above is the kinetic momentum of the system. Thus the evenised canonical momentum in this study becomes equivalent to the canonical momentum and the kinetic momentum, unlike in Ref. [2].

The single particle trajectory has been obtained in a frame in which the x and y components of the momentum are 0. For a single particle such a frame may be defined but not so for a collection of particles. Therefore the treatment above may be treated as that for a 1+1 dimensional universe only. It is non-trivial to derive as above the kinetic equation for a 3+1 dimensional universe because of the presence of spin.

III. KINETIC EQUATION USING THE WIGNER FORMALISM

We now present a more formal field theoretic derivation of the kinetic equation in a manner analogous to that adopted in Ref. [3] for the unrotated Lagrangian. As stated in the Introduction, one can obtain the kinetic equation from the equation of motion for the Wigner transform of the Wightman function. We shall closely follow the derivation in Ref. [3] for the unrotated Lagrangian. The Wightman function is defined as

$$G^{<}_{\alpha\beta}(u,v) \equiv i \left\langle \bar{\psi}_\beta(v)\psi_\alpha(u) \right\rangle . \tag{30}$$

Starting from the Dirac equation for the fermions in the electroweak plasma, one can obtain the equation of motion for the Wightman function by multiplying the Dirac equation by $i\bar{\psi}$, passing it through the differential operator and taking the expectation value.

$$[i\,\partial\!\!\!/_u - \frac{m(u)}{\hbar} + \frac{\partial_\mu\theta(u)}{2}\gamma^\mu\gamma^5]_{\rho\alpha} i \left\langle \bar{\psi}_\beta(v)\psi_\alpha(u) \right\rangle = 0 \tag{31}$$

Subsequently, performing a Wigner transform of $G^{<}(u,v)$, one obtains $\mathcal{G}^{<}(x,k)$.

$$\mathcal{G}^{<}(x,k) \equiv \int d^4r\, e^{(i/\hbar)k\cdot r} G^{<}(x+r/2, x-r/2), \tag{32}$$

where $r = u - v$. Unlike the Wightman function which is a function of two spacetime variables, $\mathcal{G}^{<}(x,k)$ is in a mixed representation, i.e., it is a function of x and k and hence can finally give a kinetic equation for particle and antiparticle distribution functions which are functions of position and momentum. $\mathcal{G}^{<}(x,k)$ satisfies

$$\left(\hat{k\!\!\!/} - \hat{m}(x) + \frac{\hbar\widehat{\partial_\mu\theta}}{2}\gamma^\mu\gamma^5 \right) \mathcal{G}^{<} = 0, \tag{33}$$

where

$$\hat{k}_\mu \equiv k_\mu + \frac{i\hbar}{2}\partial_\mu \tag{34}$$

$$\hat{m} \equiv m e^{-\frac{i\hbar}{2}\overleftarrow{\partial}_x \cdot \partial_k} \tag{35}$$

$$\hat{\partial_\mu \theta} \equiv (\partial_\mu \theta) e^{-\frac{i\hbar}{2}\overleftarrow{\partial}_x \cdot \partial_k} \tag{36}$$

(Clearly, $\hat{}$ in this section is defined differently than in the previous section.) We now assume that the bubbles are large enough that we can take the wall to be planar (and in the $x-y$ plane). Then, working in a wall frame, as in Ref. [3], in which the momentum of the particles parallel to the wall vanishes, $S^3 i\gamma^0 \mathcal{G}^< = s i\gamma^0 \mathcal{G}^<$. Therefore, since $(I - sS^3) i\gamma^0 \mathcal{G}^< = 0$, $\mathcal{G}^<(x,k)$ may be expressed, in the chiral representation, as

$$-i\gamma^0 \mathcal{G}^< = g_s^< \otimes \frac{1}{2}(I + s\sigma^3) \tag{37}$$

$$= \frac{1}{2}\left(g_0^s I + g_i^s \rho^i\right) \otimes \frac{1}{2}(I + s\sigma^3), \tag{38}$$

where $\sigma^i \equiv \rho^i$ are the Pauli matrices and $g^s_{0,i}$ are functions of k^3, x^3, t and s. [4] As discussed earlier, the Wigner function above is now for a 1+1 dimensional universe.

This decomposition allows one to reduce the 4x4 equation for the Wigner function, Eq. (33) to a 2x2 equation for $g_s^<$. Expanding the operator acting on $-i\gamma^0\mathcal{G}^<$ as a direct product of 2x2 matrices (see Ref. [10]), we get

$$[\hat{k}^0 + \frac{\hbar s}{2}\hat{\partial_3 \theta} - s\hat{k}^3\rho^3 - \hat{m}\rho^1] g_s^< = 0\,. \tag{39}$$

(Here we have used the definition of the gamma matrices as in Refs. [3, 4], which differs from the definition of γ^0 and γ^5 in Ref. [10] by a minus sign. The metric is (1,-1,-1,-1).) The corresponding equation obtained in Ref. [3] for the unrotated Lagrangian is

$$[\hat{k}^0 - s\hat{k}^3\rho^3 - \hat{m}_0\rho^1 - \hat{m}_5\rho^2] g_s^< = 0 \tag{40}$$

where

$$\hat{m}_{0,5} \equiv m_{R,I} e^{-\frac{i\hbar}{2}\overleftarrow{\partial}_x \cdot \partial_k}\,, \tag{41}$$

[4] Here we have used the definition of the direct product as in Ref. [4, 10]. The order of the two terms in the product is reversed in Ref. [3]. Also, the above decomposition of $\mathcal{G}^<(x,k)$ is as in Refs. [3, 11] which differs by a factor of 2 from the decomposition in Ref. [4].

where m_R and m_I are $|m|\cos\theta$ and $|m|\sin\theta$ respectively. Expanding $g_s^<$ in terms of I and the Pauli matrices as in Eq. (38) and multiplying Eq. (39) by I and ρ^i and taking the trace gives

$$(\hat{k}^0 + \frac{s\hbar}{2}\hat{\partial_3\theta})g_0^s - \ s\hat{k}^3 g_3^s - \ \hat{m} g_1^s \ = \ 0 \tag{42}$$
$$(\hat{k}^0 + \frac{s\hbar}{2}\hat{\partial_3\theta})g_3^s - \ s\hat{k}^3 g_0^s - i\hat{m} g_2^s \ = \ 0 \tag{43}$$
$$(\hat{k}^0 + \frac{s\hbar}{2}\hat{\partial_3\theta})g_1^s + is\hat{k}^3 g_2^s - \ \hat{m} g_0^s \ = \ 0 \tag{44}$$
$$(\hat{k}^0 + \frac{s\hbar}{2}\hat{\partial_3\theta})g_2^s - is\hat{k}^3 g_1^s + i\hat{m} g_3^s \ = \ 0. \tag{45}$$

These equations are similar to Eqs. (2.16)-(2.19) of Ref. [3] but with $\hat{k}^0 \to \hat{k}^0 + \frac{s\hbar}{2}\hat{\partial_3\theta}$, $\hat{m}_0 \to \hat{m}$ and $\hat{m}_5 \to 0$. One can independently set the real and imaginary parts of these equations to 0. The imaginary parts contain time derivatives and so provide the kinetic equations. The real parts provide constraint equations. Assuming that m and $\partial_3\theta$ vary slowly in the Higgs bubble wall we may expand $\hat{m}$ and $\hat{\partial_3\theta}$ to $O(\hbar)$ as

$$\hat{m} \ = \ m + \frac{i\hbar}{2} m' \partial_{k_z} \tag{46}$$
$$\hat{\partial_3\theta} \ = \ \theta' + \frac{i\hbar}{2}\theta'' \partial_{k_z} \, . \tag{47}$$

Henceforth, we define $z \equiv x^3$ and $k_z \equiv k^3$, and $'$ denotes $\partial/\partial x^3 = \partial/\partial z$ and $\partial_{k_z} = \partial/\partial k^3$. $\overleftarrow{\partial}_x \cdot \partial_k = -\overleftarrow{\partial}_z \cdot \partial_{k_z}$. The constraint equations then are

$$(k^0 + \frac{s\hbar}{2}\theta')g_0^s - s k_z g_3^s - m g_1^s = 0 \tag{48}$$
$$(k^0 + \frac{s\hbar}{2}\theta')g_3^s - s k_z g_0^s + \frac{\hbar}{2} m' \partial_{k_z} g_2^s = 0 \tag{49}$$
$$(k^0 + \frac{s\hbar}{2}\theta')g_1^s + \frac{s\hbar}{2}\partial_z g_2^s - m g_0^s = 0 \tag{50}$$
$$(k^0 + \frac{s\hbar}{2}\theta')g_2^s - \frac{s\hbar}{2}\partial_z g_1^s - \frac{\hbar}{2} m' \partial_{k_z} g_3^s = 0. \tag{51}$$

The above equations imply relations between g_0^s and g_i^s. Using Eqs. (49,50,51) iteratively to $O(\hbar)$ we express the g_i^s in terms of g_0^s. Substituting this in Eq. (48) gives

$$\left((k^0 + \frac{\hbar s}{2}\theta')^2 - k_z^2 - m^2 \right) g_0^s = 0 \, , \, . \tag{52}$$

This implies

$$\Omega_s^2 \ \equiv \ (k^0 + \frac{\hbar s}{2}\theta')^2 - k_z^2 - m^2 = \ 0, \tag{53}$$

whose solutions are $k^0 = \pm\omega_{s\pm}(z, k_z)$ with

$$\omega_{s\pm}(z, k_z) = \omega_0 \mp \frac{\hbar s}{2}\theta' \,, \qquad \omega_0 = \sqrt{k_z^2 + m^2} \tag{54}$$

$\omega_{s+}(z, k_z)$ and $\omega_{(-s)-}(z, -k_z)$ will represent the energy of particles/antiparticles as they pass through the Higgs bubble wall. This agrees with Eq. (24).

The kinetic equation for g_0^s is

$$\partial_t g_0^s + \frac{\hbar s}{2}\theta'' \partial_{k_z} g_0^s + s\partial_z g_3^s - m'\partial_{k_z} g_1^s = 0 \,. \tag{55}$$

Using the constraint equations to re-express g_i^s in terms of g_0^s, the kinetic equation for g_0^s implies

$$\partial_t g_0^s + k_z \partial_z \left(\frac{g_0^s}{k^0 + \frac{\hbar s \theta'}{2}} \right) - \frac{1}{2}{m^2}' \partial_{k_z} \left(\frac{g_0^s}{k^0 + \frac{\hbar s \theta'}{2}} \right) + \frac{\hbar s}{2}\theta'' \partial_{k_z} g_0^s = 0 \,. \tag{56}$$

Analogous to Ref. [11], Eq. (52) implies that g_0^s may be written as

$$g_0^s = 4\pi N |k^0| \delta(\Omega_s^2) \,. \tag{57}$$

However, for reasons discussed below, we define

$$g_0^s = 4\pi N |k^0 + \hbar s \theta'/2| \delta(\Omega_s^2) \tag{58}$$

$$= \sum_{\pm} 4\pi \, N \frac{|k^0 + \hbar s\theta'/2|}{2\omega_0} \, \delta(k^0 \mp \omega_{s\pm}) \,. \tag{59}$$

N is a function of k^0, k_z, z, t and spin s.

One now substitutes for g_0^s from Eq. (59) in the kinetic equation and integrates over positive and negative energies separately to get

$$\partial_t f_{s\pm} + \left(\frac{k_z}{\omega_0} \right) \partial_z f_{s\pm} + \left(-\frac{{m^2}'}{2\omega_0} + \frac{\hbar s}{2}\theta'' \right) \partial_{k_z} f_{s\pm} = 0. \tag{60}$$

where $f_{s\pm}(k_z, z, t)$ are functions given by

$$f_{s+} \equiv N(\omega_{s+}, k_z, z, t) \tag{61}$$

$$f_{s-} \equiv 1 - N(-\omega_{(-s)-}, -k_z, z, t) \tag{62}$$

If we can identify $f_{s\pm}$ with the particle and antiparticle distribution functions [5], the above equations have the form of the classical kinetic equation, with quantum corrections in the coefficients of the derivatives of $f_{s\pm}$.

[5] See footnote [2].

The identification of the particle and antiparticle distribution functions can be made in two ways. One can argue that the distribution function is that quantity which gives the number density on integration over momenta. Defining the current as

$$j^\mu(x) = \left\langle \bar{\psi}(x)\gamma^\mu\psi(x)\right\rangle \tag{63}$$

we rewrite it as in Sec. 4.1 of Ref. [5] as

$$j^\mu(x) = -\mathrm{Tr}[\gamma^\mu iG^<(x,x)] = -\int \frac{d^2k}{(2\pi)^2}\mathrm{Tr}\gamma^\mu i\mathcal{G}^<(k,x) \tag{64}$$

Substituting for the Wigner function from Eq. (38), and using $\mathrm{Tr}A\otimes B = \mathrm{Tr}A\,\mathrm{Tr}B$ and the tracelessness of the Pauli matrices, we get

$$j^0 = \sum_{s=\pm1}\int \frac{d^2k}{(2\pi)^2} g_0^s(k^0,k_z,z,t)\,. \tag{65}$$

If we use the decomposition in Eq. (59) for g_0^s we find that we can write j^0 as $j_+^0 - j_-^0$ where

$$j_\pm^0 = \sum_{s=\pm1} j_{s\pm}^0 = \sum_{s=\pm1}\int \frac{dk_z}{(2\pi)} f_{s\pm}\,. \tag{66}$$

(We have ignored the vacuum contribution to j^0.) Thus if we define particle and antiparticle number densities (per unit length) to be $j_\pm^0$ it seems appropriate to identify $f_{s\pm}$ with particle and antiparticle distribution functions. Alternatively, one can argue that the quantum corrected kinetic equation should have the same form as the classical kinetic equation, i.e., it should contain only derivatives of the distribution function and not terms proportional to the distribution function. Since the equation for $f_{s\pm}$ satisfies this criterion it again seems correct to identify $f_{s\pm}$ with particle and antiparticle distribution functions. However, note that if we had used Eq. (57) as the decomposition for g_0^s neither of the above criteria would be satisfied. This is why we chose the decomposition in Eq. (59). We further note that with the $f_{s\pm}$ identified as particle and antiparticle distribution functions Eq. (60) above agrees with the kinetic equation obtained in Sec. II.

We briefly comment on the application of the above ideas to the exposition in Refs. [3, 4, 5] for the unrotated Lagrangian. While f_{s+} in these works satisfies the second criterion above, i.e., the quantum corrected kinetic equation has the form of the classical kinetic equation, the quantity whose integral over momentum gives j_+^0 are f_{s+}/Z_{s+}, i.e.,

$$j_{s+}^0 = \int \frac{dk_z}{2\pi} f_{s+}/Z_{s+}\,, \tag{67}$$

where

$$Z_{s+} \equiv \frac{1}{2\omega_{s+}} |\partial_{k^0} \Omega_s^2|_{k^0=\omega_{s+}} \,. \tag{68}$$

(See Eq. (2.41) of Ref. [3] but with the normalisation of g_0^s as in Ref. [11]. [6]) If one rewrites the kinetic equation in terms of the variable f_{s+}/Z_{s+} one gets an additional term in the kinetic equation.

But let us now consider the quantity which gives the number density of particles on integration over 4-momentum, or rather, 2-momentum in our 1+1 dimensional case, after multiplying with kinetic energy and a delta function containing the mass shell condition, i.e.,

$$j_{s+}^0 = \int \frac{dk_z \, dk^0}{(2\pi)^2} \, 2K^0 \, 2\pi\delta(\Omega^2) \, F_s(x,k) \,, \tag{69}$$

where K^0 is the kinetic energy. When converting $\delta(\Omega^2)$ to $\delta(k^0 - \omega_{s+})$ one ultimately gets a factor of Z_{s+} in the denominator. (For the unrotated Lagrangian, the kinetic energy K^0 equals the total energy ω_{s+}, as evidenced by the definition of the kinetic momentum in Eq. (2.14) of Ref. [8] as ωv_g, where ω is the total energy and v_g is the velocity.) Now if we choose to define the particle distribtuion functions as $F_s(x,k)$ with k^0 replaced by ω_{s+}, i.e., in terms of the function in the integrand of j_{s+}^0 when expressed as an integral over k_z and k^0 as in Eq. (69) rather than as an integral over only k_z as in Eq. (67), then we can identify f_{s+}, rather than f_{s+}/Z_{s+}, with the particle distribution function and one also has a 'standard' form for the kinetic equation for this particle distribution function. Therefore this prompts us to modify our first scheme of identification of the particle distribution function. It does not affect the case of the rotated Lagrangian.

The transport equations relevant for electroweak baryogenesis for the unrotated Lagrangian are obtained in Eqs. (5.2), and (5.8) and (5.9) of Ref. [3] from the kinetic equations for g_0^s and g_3^s, which agree with taking the zeroth and first moments of the kinetic equation for f_{s+}, i.e., by integrating the kinetic equation over k_z after multiplying by 1 and the velocity v. For the rotated Lagrangian considered in this article we obtain the kinetic equation using both methods and they agree (for us, $v \equiv k_z/\omega_0$). The equations we obtain are

$$\partial_t n_{s\pm} + \partial_z (n_{s\pm} u_{s\pm}) \;=\; 0 \tag{70}$$

[6] The Z factor is absent in Eq. (4.10) of Ref. [5] because the quantity δf_s^v in the numerator is already $O(\hbar)$.

$$\partial_t \left(n_{s\pm} u_{s\pm}\right) + \partial_z \left(n_{s\pm}\langle v_{s\pm}^2\rangle\right) + \frac{1}{2} m^{2\prime} \mathcal{I}_{2s\pm} - \frac{1}{2} s(m^2\theta')' \mathcal{I}_{3s\pm} \;=\; 0 \tag{71}$$

where

$$n_{s\pm} \;=\; \int \frac{dk_z}{2\pi} f_{s\pm} \tag{72}$$

$$n_{s+}\langle v_{s+}^p\rangle \;\equiv\; \int \frac{dk_z}{2\pi} \left(\frac{k_z}{\omega_0}\right)^p f_{s\pm}\,, \tag{73}$$

and $u_{s\pm} \equiv \langle v_{s\pm}\rangle$, and

$$\mathcal{I}_{ps\pm} = \int \frac{dk_z}{2\pi} \frac{f_{s\pm}}{\omega^p_{(\pm s)\pm}} \quad (p = 2, 3)\,. \tag{74}$$

It is interesting to note that the form of the above transport equations for the rotated Lagrangian are identical to those in Ref. [3] for the unrotated Lagrangian (if one amends Ref. [3] to identify antiparticles of spin s with negative energy particles of spin $-s$).

IV. CONCLUSION

In conclusion, we have obtained the energy relation and the kinetic equation for fermions and antifermions interacting with the Higgs bubble wall during the electroweak phase transition using the method of evenisation and the Wigner formalism. Our results for the velocity and force on the particle/antiparticle as they pass through the Higgs wall are the same using both methods. This indicates that evenisation is indeed a reliable method to investigate the quantum corrections to the velocity and force acting on particles and antiparticles as they traverse the bubble wall during the electroweak phase transition.

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Appendix

We present below the derivations of the various evenised expressions quoted in the text. Standard formulae that we shall use are $\hat{\beta}^2 = -(\hat{\gamma}^0\hat{\gamma}^5)^2 = (\hat{\alpha}^3)^2 = 1$. The symbols square brackets are used below for both commutators and evenised operators but the difference is clear from the context.

Evenised $\hat{\beta}$, $\hat{\alpha}^3$ and $\hat{\gamma}^0\hat{\gamma}^5$ to $O(\hbar^0)$:

To $O(\hbar^0)$,

$$\begin{aligned}
2[\hat{\beta}] &= \hat{\beta} + \hat{\Lambda}_0 \hat{\beta} \hat{\Lambda}_0 \\
&= \hat{\beta} + \frac{\hat{\alpha}^3 p^3 + \hat{\beta}\hat{m}}{E_0} \hat{\beta} \hat{\Lambda}_0 \\
&= \hat{\beta} + \hat{\beta} \frac{-\hat{\alpha}^3 p^3 + \hat{\beta}\hat{m}}{E_0} \hat{\Lambda}_0 \\
&= \hat{\beta} + \hat{\beta} \frac{-\hat{\Lambda}_0 E_0 + 2\hat{\beta}\hat{m}}{E_0} \hat{\Lambda}_0 \\
&= \frac{2\hat{m}}{E_0} \hat{\Lambda}_0 \,,
\end{aligned} \tag{75}$$

where we have used $\hat{\Lambda}_0^2 = 1 + O(\hbar)$ in the last equality. Since the lhs is even and the sign operator on the rhs is even, $m(\hat{z})$ may be replaced by $[m(\hat{z})]$, which, as explained in the text, is equal to $m([\hat{z}])$ to $O(\hbar)$. Therefore

$$[\hat{\beta}] = \frac{m([\hat{z}])}{E_0} \hat{\Lambda}_0 + O(\hbar) \,. \tag{76}$$

Similarly, to $O(\hbar^0)$,

$$\begin{aligned}
2[\hat{\alpha}^3] &= \hat{\alpha}^3 + \hat{\Lambda}_0 \hat{\alpha}^3 \hat{\Lambda}_0 \\
&= \hat{\alpha}^3 + \frac{\hat{\alpha}^3 p^3 + \hat{\beta}\hat{m}}{E_0} \hat{\alpha}^3 \hat{\Lambda}_0 \\
&= \hat{\alpha}^3 + \hat{\alpha}^3 \frac{\hat{\alpha}^3 p^3 - \hat{\beta}\hat{m}}{E_0} \hat{\Lambda}_0 \\
&= \hat{\alpha}^3 + \hat{\alpha}^3 \frac{-\hat{\Lambda}_0 E_0 + 2\hat{\alpha}^3 p^3}{E_0} \hat{\Lambda}_0 \\
&= \frac{2\hat{p}^3}{E_0} \hat{\Lambda}_0 \; = \frac{2\hat{p}_z}{E_0} \hat{\Lambda}_0 \,,
\end{aligned} \tag{77}$$

Again, since the lhs is even and the sign operator on the rhs is even, $\hat{p}_z$ may be replaced by $[\hat{p}_z]$. Therefore

$$[\hat{\alpha}^3] = \frac{[\hat{p}_z]}{E_0} \hat{\Lambda}_0 + O(\hbar) \,. \tag{78}$$

Analogously,

$$\begin{aligned}
2[\hat{\gamma}^0 \hat{\gamma}^5] &= \hat{\gamma}^0 \hat{\gamma}^5 + \hat{\Lambda}_0 \hat{\gamma}^0 \hat{\gamma}^5 \hat{\Lambda}_0 \\
&= \hat{\gamma}^0 \hat{\gamma}^5 - \hat{\gamma}^0 \hat{\gamma}^5 \hat{\Lambda}_0 \hat{\Lambda}_0 \\
&= 0 \,.
\end{aligned} \tag{79}$$

Therefore,

$$[\hat{\gamma}^0 \hat{\gamma}^5] = O(\hbar) \,. \tag{80}$$

Evenised expressions for the velocity and the force to $O(\hbar)$:

Since we wish to work till $O(\hbar)$ we now use the sign operator $\hat{\Lambda}$ defined as

$$\hat{\Lambda} = \frac{\hat{H}}{E} \tag{81}$$

with E defined to $O(\hbar)$ as in Eq. (24). Now $d\hat{z}/dt = -(i/\hbar)[\hat{z}, \hat{H}] = \hat{\alpha}^3$. Then

$$\begin{aligned} 2[\hat{\alpha}^3] &= \hat{\alpha}^3 + \hat{\Lambda}\hat{\alpha}^3\hat{\Lambda} \\ &= \hat{\alpha}^3 + \hat{\alpha}^3 \frac{-\hat{\Lambda}E + 2\hat{\alpha}^3\hat{p}^3 - \hbar\hat{\theta}'\hat{S}^3}{E}\hat{\Lambda} \\ &= \frac{2\hat{p}^3 - \hbar\hat{\theta}'\hat{\alpha}^3\hat{S}^3}{E}\hat{\Lambda} = \frac{2\hat{p}_z}{E}\hat{\Lambda} - \frac{\hbar\hat{\theta}'\hat{\alpha}^3\hat{S}^3}{E}\hat{\Lambda}\,. \end{aligned} \tag{82}$$

Since the lhs of the above equation and the sign operator on the rhs are even we may replace $\hat{p}_z$ by $[\hat{p}_z]$. Since $\hat{S}^3$ is also even, we may similarly replace $\hat{\theta}'\hat{\alpha}^3$ by $[\hat{\theta}'\hat{\alpha}^3]$. Since $\{\hat{\theta}'\}$ is proportional to $[\hat{\theta}', \hat{\Lambda}] \sim \hbar$ we may rewrite $[\hat{\theta}'\hat{\alpha}^3]$ as $[\hat{\theta}'][\hat{\alpha}^3]$. As discussed in the text for $m(\hat{z})$, $[\hat{\theta}'] = \theta'([\hat{z}]) + O(\hbar^2)$. Further we may use the expression for $[\hat{\alpha}^3]$ obtained above in Eq. (78) upto $O(\hbar^0)$ in the rhs of the above equation. Therefore we get, to $O(\hbar)$,

$$[d\hat{z}/dt] = \left(\frac{[\hat{p}_z]}{E} - \frac{\hbar\hat{\theta}'[\hat{p}_z]\hat{S}^3\hat{\Lambda}_0}{2E_0^2}\right)\hat{\Lambda}\,. \tag{83}$$

The force $d\hat{p}_z/dt = -(i/\hbar)[\hat{p}_z, \hat{H}] = -\hat{\beta}\hat{m}' + \hbar\hat{\theta}''\hat{S}^3/2$. Then

$$\begin{aligned} 2[\hat{\beta}\hat{m}'] &= \hat{\beta}\hat{m}' + \hat{\Lambda}\hat{\beta}\hat{m}'\hat{\Lambda} \\ &= \hat{\beta}\hat{m}' + \hat{\beta}\hat{m}' \frac{-\hat{\Lambda}E + 2\hat{\beta}\hat{m} - \hbar\hat{\theta}'\hat{S}^3}{E}\hat{\Lambda} - \hat{\beta}\hat{\alpha}^3\frac{[\hat{p}_z, \hat{m}']}{E}\hat{\Lambda} \\ &= \frac{2\hat{m}'\hat{m} - \hbar\hat{\beta}\hat{m}'\hat{\theta}'\hat{S}^3 + i\hbar\hat{\beta}\hat{\alpha}^3\hat{m}''}{E}\hat{\Lambda}\,. \end{aligned} \tag{84}$$

Again, we may replace $\hat{m}'\hat{m}$ by $[\hat{m}'\hat{m}]$ as both sides of the equation should be even. As $\{\hat{m}'\}\{\hat{m}\}$ is $O(\hbar^2)$, Eq. (7) implies that this reduces to $[m'(\hat{z})][m(\hat{z})]$. This may further be rewritten as $m'([\hat{z}])m([\hat{z}])$. For the second term in the numerator, we replace $\hat{\beta}\hat{m}'\hat{\theta}'\hat{S}^3$ by $[\hat{\beta}\hat{m}'\hat{\theta}'\hat{S}^3] = [\hat{\beta}\hat{m}'\hat{\theta}']\hat{S}^3$. This may be rewritten as $[\hat{\beta}\hat{m}'][\hat{\theta}']\hat{S}^3$. As above, we replace $[\theta'(\hat{z})]$ by $\theta'([\hat{z}])$. Since, by Eq. (6), $\{\hat{m}'\}$ is $O(\hbar)$, $[\hat{\beta}\hat{m}']$ can be rewritten as $[\hat{\beta}][\hat{m}']$, and $[\hat{\beta}]$ may be replaced by the $O(\hbar^0)$ expression in Eq. (76) and $[\hat{m}']$ by $\hat{m}'([\hat{z}])$. For the third term, we replace $\hat{\beta}\hat{\alpha}^3\hat{m}''$ by $[\hat{\beta}\hat{\alpha}^3\hat{m}'']$. Since this term is already of $O(\hbar)$ we can write it as $[\hat{\beta}\hat{\alpha}^3][m(\hat{z})'']$ and further as $[\hat{\beta}\hat{\alpha}^3]m''([\hat{z}])$. Now $\hat{\beta}\hat{\alpha}^3 = \hat{S}^3\hat{\gamma}^0\hat{\gamma}^5$ and since $\{\hat{S}^3\} = 0$, $[\hat{\beta}\hat{\alpha}^3] = \hat{S}^3[\hat{\gamma}^0\hat{\gamma}^5] = O(\hbar)$. Therefore, this term is of $O(\hbar^2)$ and we may ignore it. Thus

$$[\hat{\beta}\hat{m}'] = \left(\frac{m([\hat{z}])^{2\prime}}{2E} - \frac{\hbar m([\hat{z}])^{2\prime}\theta'([\hat{z}])\hat{\Lambda}_0\hat{S}^3}{4E_0^2}\right)\hat{\Lambda}\,. \tag{85}$$

Therefore

$$
\begin{aligned}
[d\hat{p}_z/dt] &= \left(-\frac{\hat{m}^{2\prime}}{2E} + \frac{\hbar\hat{m}^{2\prime}\hat{\theta}'\hat{\Lambda}_0\hat{S}^3}{4E_0^2}\right)\hat{\Lambda} + \frac{\hbar\hat{\theta}''\hat{S}^3}{2} && (86)\\
&= \left(-\frac{\hat{m}^{2\prime}}{2E} + \frac{\hbar\hat{m}^{2\prime}\hat{\theta}'\hat{\Lambda}_0\hat{S}^3}{4E_0^2} + \frac{\hbar\hat{\theta}''\hat{S}^3}{2}\hat{\Lambda}_0\right)\hat{\Lambda}\,. && (87)
\end{aligned}
$$

$\hat{m}$ and $\hat{\theta}$ above are functions of $[\hat{z}]$.

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[1] W. Greiner, *Relativistic Quantum Mechanics*, Springer-Verlag, Berlin, Third Edition (2000) (Ch. 2).

[2] J. R. Bhatt and R. Rangarajan, to appear in Phys. Rev. D [arXiv:hep-ph/0404284].

[3] K. Kainulainen, T. Prokopec, M. G. Schmidt and S. Weinstock, JHEP 0106 (2001) 31 [arXiv:hep-ph/0105295].

[4] T. Prokopec, M. G. Schmidt and S. Weinstock, Ann. Phys. 314 (2004) 208 [arXiv:hep-ph/0312110].

[5] T. Prokopec, M. G. Schmidt and S. Weinstock, Ann. Phys. 314 (2004) 267 [arXiv:hep-ph/0406140].

[6] M. Joyce, T. Prokopec and N. Turok, Phys. Rev. Lett. **75** (1995) 1695; erratum-ibid **75** (1995) 3375 [arXiv:hep-ph/9408339].

[7] M. Joyce, T. Prokopec and N. Turok, Phys. Rev. D **53**, (1996) 2958 [arXiv:hep-ph/9410282].

[8] J. M. Cline, M. Joyce and K. Kainulainen, JHEP 0007 (2000) 018 [arXiv:hep-ph/0006119]; erratum, arXiv:hep-ph/0110031.

[9] D. Vasak, M. Gyulassy and H.-T. Elze, Annals of Physics 173 (1987) 462.

[10] C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, McGraw-Hill, (1980), Appendix A-2.

[11] K. Kainulainen, T. Prokopec, M. G. Schmidt and S. Weinstock, Phys. Rev. D **66** (2002) 043502 [arXiv:hep-ph/0202177].